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The structure of smooth algebras in Kapranov's framework for noncommutative geometry

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Abstract

In [M. Kapranov, Noncommutative geometry based on commutator expansions, J. Reine Angew. Math. 505 (1998) 73–118] a theory of noncommutative algebraic varieties was proposed. Here we prove a structure theorem for the noncommutative coordinate rings of affine open subsets of such of those varieties which are smooth (Theorem 3.4). The theorem describes the local ring of a point as a truncation of a quantization of the enveloping Poisson algebra of a smooth commutative local algebra. An explicit description of this quantization is given in Theorem 2.5. A description of the A-module structure of the Poisson envelope of a smooth commutative algebra A was given in loc. cit., Theorem 4.1.3. However the proof given in loc. cit. has a gap. We fix this gap for A local (Theorem 1.4) and prove a weaker global result (Theorem 1.6).

Keywords: Commutator filtration; Poisson algebra; d-smooth algebra

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0. Introduction

We consider associative, unital algebras over a fixed algebraically closed field k of characteristic zero. If R is an algebra, then the *commutator filtration* of R is defined as

$$F_0R = R, \qquad F_{n+1}R := \sum_{p=1}^{n} F_pRF_{n+1-p}R + \sum_{p=0}^{n}\langle [F_pR, F_{n-p}R]\rangle. \tag{1}$$

(One checks that this filtration is the same as that called NC-filtration in [2].) By definition, FR is the smallest of all descending filtrations $\mathcal{G}$ with $\mathcal{G}_p\mathcal{G}_q \subset \mathcal{G}_{p+q}$, $[\mathcal{G}_p, \mathcal{G}_q] \subset \mathcal{G}_{p+q+1}$. An algebra R is *nilcommutative of order* d if $F_{d+1}R = 0$. Thus the nilcommutative algebras of order 0 are just the commutative algebras. We write NC_d for the category of nilcommutative algebras of order d and algebra homomorphisms and set $NC = \bigcup_{d=0}^{\infty} NC_d$. An algebra R is called d*-formally smooth* if $R \in NC_d$ and if $\hom_{NC_d}(R, \,)$ maps surjections with nilpotent kernel to surjections, and is d*-smooth* if it is d-formally smooth and if the commutative algebra $A = R/F_1R$ is essentially of finite type. For example a 0-smooth algebra is the same thing as a smooth commutative algebra. For the remainder of this section A will be a fixed smooth commutative algebra. It was shown in [2, 1.6.1] that there exists a tower of surjective homomorphisms:

$$\cdots \twoheadrightarrow R_{d+1} \twoheadrightarrow R_d \twoheadrightarrow \cdots \twoheadrightarrow R_2 \twoheadrightarrow R_1 \twoheadrightarrow R_0 = A$$

such that R_d is d-smooth and $R_d/F_dR = R_{d-1}$ ($d \geqslant 1$). Moreover, it is shown in loc. cit. that such a tower is unique up to noncanonical isomorphism. Kapranov further develops a theory of nilcommutative d-smooth algebraic varieties based on this 'affine' construction. In this paper we focus on the affine part of Kapranov's work. We study the structure of the algebras R_d and of their associated graded Poisson algebras $GR_d := \bigoplus_{n=0}^{d} F_nR_d/F_{n+1}R_d$. A characterization of GR_d was given in [2, 4.2.1]. It was shown that there is an isomorphism

$$PA/P_{>d}A \xrightarrow{\sim} GR_d. \tag{2}$$

Here $P : (\!(\mathrm{Comm})\!) \to (\!(\mathrm{Poiss})\!)$ is left adjoint to the forgetful functor which associates to a Poisson algebra its underlying commutative algebra. The algebra PA turns out to be graded, and $P_{>d}A = \bigoplus_{n>d} P_nA$. The map (2) is canonical, and comes from the adjointness property of P; if $R \in NC_d$ is any algebra with $G_0R = R/F_1R = A$, then the adjunction map $PA \to GR$ induced by $A \cong G_0R$ factors through $PA/P_{>d}A$ obtaining (2). Here we prove a converse of Kapranov's result. We show that if $R_d \in NC_d$ is any algebra with $R_d/F_1R_d = A$ and such that the canonical map (2) is an isomorphism, then R_d is d-smooth (Theorem 3.4). This means that to give a d-smooth algebra R with $R/F_1R = A$ is the same thing as to give an associative multiplication

$$\phi = \sum_{r=0}^{d} \phi_r : PA/P_{>d}A \otimes P/P_{>d}A \to PA/P_{>d}A \tag{3}$$

with ϕ_r homogeneous of degree r. For $(A, \mathcal{M})$ local, we give (Theorem 2.5) a canonical construction which produces an associative product

$$B^X(\hbar) = \sum_{p=0}^{\infty} B_p^X \hbar^p : PA \otimes PA[\![\hbar]\!] \to PA[\![\hbar]\!] \tag{4}$$

for each regular system of parameters $X = \{x_1, \dots, x_n\} \subset \mathcal{M}$, with B_p^X homogeneous of degree p and a bidifferential operator of order $\leqslant p$. It turns out that, modulo $P_{>d}A$, the evaluated series $B^X(1)$ is a finite sum, and gives a product ϕ satisfying the requirements of (3). We use this product to give a local characterization of R_d (Theorem 3.4). The construction of the product (4) uses a local isomorphism of A-modules

$$PA \cong S^A L_+^A \Omega_A^1 \quad (n \geqslant 0). \tag{5}$$

Between PA and the A-symmetric algebra of the Lie subalgebra

$$L_+^A \Omega_A^1 = \left[L^A \Omega_A^1, L^A \Omega_A^1\right]$$

of the free A-Lie algebra $L^A \Omega_A^1$ generated by the module of Kähler differentials. Theorem 4.1.3 of [2] states that there is a global isomorphism as that of (5); there is however a gap in the proof. The gap is explained in Section 1 below, where it is also shown how it is fixed for A local (Theorem 1.4). I do not know whether (5) still holds globally. A weaker version of (5) which holds globally is proved in Theorem 1.6; it establishes that PA carries a filtration such that the associated graded module is (globally) isomorphic to the right-hand side of (5).

When I explained to Kapranov the gap in his proof of (5), and told him the gap could be fixed locally, he suggested that a weaker version along the lines of that presented here (Theorem 1.6) should hold globally. I am thankful to him for this suggestion.

The remainder of this paper is organized as follows. In Section 1 we recall in some detail the construction of the Poisson algebra PA (which we call the Poisson envelope of A), explain the gap in Kapranov's proof of (5), and prove it in the local case (Theorem 1.4). The section ends with the weaker version of (5) which holds globally (Theorem 1.6). Section 2 is devoted to the construction of the product (4) (Theorem 2.5). The results of this section can be seen as the generalization to general local smooth algebras of those obtained by Kapranov for localizations of polynomial rings [2, §3]. Our approach is however different from that of [2]. In loc. cit. the Feynmann–Maslov calculus was used to describe the product of elements in the tensor algebra on a finite dimensional vectorspace V in terms of a specific ordered basis. Instead we use the coordinate free approach of [1], where explicit formulas for this product were obtained for not necessarily finite dimensional vectorspaces V. The same formulas apply to the quantized product (4). In Section 3 we prove (Theorem 3.4) that, for B^X as in (4), an algebra $R \in NC_d$ is

(i) d-smooth $\Leftrightarrow$
(ii) $A := R/F_1R$ is smooth commutative and (2) is an isomorphism $\Leftrightarrow$

(iii) R is locally isomorphic to $(PA/P_{>d}A, B^X(1))$ for some regular system of parameters X.

Part (i) $\Rightarrow$ (ii) of this was proven by Kapranov in [2, 4.2.1]; we give a new proof.

1. The Poisson envelope of a commutative algebra

1.0. Two gradings in the symmetric algebra of a free Lie algebra

If V is a vectorspace, we write TV for the tensor algebra and $LV \subset TV$ for the Lie subalgebra it generates. For $V = \bigoplus_{x\in X} kx$—the free vectorspace on a set X—LV is the free Lie algebra on X. The symmetric algebra $S\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$ is viewed as a Poisson algebra via the Poisson bracket $\{,\}$ induced by the Lie bracket $[,]$ of $\mathfrak{g}$. For example,

$$\operatorname{Poiss} V := SLV$$

is a free Poisson algebra. Fix a vectorspace V and set $L = LV$. We have $L = \bigoplus_{n\geqslant 0} L_n$, where

$$L_0 = V, \qquad L_{n+1} = [L_0, L_n] \quad (n \geqslant 0). \tag{6}$$

Note our grading is the usual one—as defined, for example, in [3, LA, Chapter IV]—shifted down one degree. Put

$$|l|_* = n \quad \text{if } l \in L_n \ (n \geqslant 0). \tag{7}$$

This grading induces one in the symmetric algebra $S = SL$; we write S_n for its homogeneous part of degree n. Note that S_n is not the same thing as the nth symmetric power $S^n = S^n L$. The latter is the homogenous part of degree n with respect to a different grading, namely that given by

$$|l|^* = 1 \quad \text{if } l \in L. \tag{8}$$

Put

$$L_+ = \bigoplus_{n\geqslant 1} L_n.$$

We have $S_0 L = SV$ and for $n \geqslant 1$,

$$\begin{gathered} S_n L = SV \otimes S_n L_+, \\ S_n L_+ = \bigoplus_{r\geqslant 1} \bigoplus_{\substack{0<i_1<\cdots<i_r \\ p_1 i_1+\cdots+p_r i_r = n \\ p_1,\ldots,p_r>0}} S^{p_1} L_{i_1} \otimes \cdots \otimes S^{p_r} L_{i_r}. \end{gathered} \tag{9}$$

1.1. Poisson ideals

A *Poisson ideal* in a Poisson algebra P is a subspace $I \subset P$ which is an ideal for both the associative and the Lie algebra structures. If $Y \subset P$ is a subset, then we put $\langle Y\rangle$ and $\langle\!\langle Y\rangle\!\rangle$ for the smallest ideal and the smallest Poisson ideal containing Y. By definition $\langle Y\rangle \subset \langle\!\langle Y\rangle\!\rangle$. In fact $\langle\!\langle Y\rangle\!\rangle$ is generated as an ideal by the elements of Y and by those of the form

$$\bigl\{a_1, \bigl\{a_2, \dots, \{a_n, y\}\dots\bigr\}\bigr\} \quad n \geqslant 1,\ a_i \in P,\ y \in Y\ (1 \leqslant i \leqslant n).$$

Furthermore, one checks that if $X \subset P$ generates P as a Poisson algebra then for

$$g_i(x_1, \dots, x_n; y) := \bigl\{x_1, \bigl\{x_2, \dots, \bigl\{x_i, \bigl\{y, \bigl\{x_{i+1}, \dots, \{x_{n-1}, x_n\}\dots\bigr\}\bigr\}\bigr\}\dots\bigr\}\bigr\} \tag{10}$$

we have

$$\langle\!\langle Y\rangle\!\rangle = \left\langle Y \cup \bigcup_{n=1}^{\infty} \bigl\{g_i(x_1, \dots, x_n; y):\ 0 \leqslant i \leqslant n,\ x_i \in X,\ y \in Y\bigr\}\right\rangle. \tag{11}$$

1.2. Poisson envelope

Let A be a commutative algebra, SA the symmetric algebra on its underlying vectorspace, $SA \twoheadrightarrow A$ the canonical projection, IA its kernel. The *Poisson envelope* of A is

$$PA := \frac{SLA}{\langle\!\langle IA\rangle\!\rangle}.$$

The inclusion $A = SA/IA \subset PA$ has the following universal property. If P is a Poisson algebra and $f : A \to P$ is a homomorphism of commutative algebras, then there is a unique Poisson homomorphism $PA \to P$ which extends f. In other words, $A \mapsto PA$ is left adjoint to the forgetful functor $(\!(\text{Poiss})\!) \to (\!(\text{Comm})\!)$ from Poisson to commutative algebras. One checks that if $A = SV/I$ is any presentation of A as a quotient of a symmetric algebra, then $SLV/\langle\!\langle I\rangle\!\rangle$ has the same universal property as and is therefore isomorphic to PA. In particular,

$$PSV = \operatorname{Poiss} V.$$

It follows from (11) that if $I \subset SV$ is as above then $\langle\!\langle I\rangle\!\rangle \subset SLV$ is homogeneous for the grading (7), whence PA inherits a grading:

$$PA = \bigoplus_{n\geqslant 0} P_n A.$$

For example,

$$P_n SV = S_n LV \quad (n \geqslant 0). \tag{12}$$

In particular,

$$P_1SV = SV \otimes L_1V = SV \otimes \Lambda^2 V = \Omega^2_{SV} \tag{13}$$

is the module of 2-differential forms. If follows from (13) and (11) that for every commutative algebra A,

$$P_1A = \frac{S_1(LA)}{IAS_1(LA) + \langle\{A, IA\}\rangle} = \frac{\Omega^2_{SA}}{IA\Omega^2_{SA} + \Omega^1_A \wedge dIA} = \Omega^2_A.$$

Under this isomorphism,

$$\{a, b\} = da \wedge db \in P_1A.$$

For another interpretation of P_1A consider the analogue L^A of the functor L for A-modules and A-Lie algebras. If M is an A-module, then L^AM carries a grading defined exactly as in (6). We have $L^A_0M = M$, $L^A_1M = \Lambda^2M$, and in particular,

$$L^A_1\Omega^1_A = \Omega^2_A = P_1A. \tag{14}$$

Theorem 4.1.3 of [2] says that a generalization of (14) holds for smooth algebras. Namely it is asserted that for every $n \geqslant 0$, P_nA is isomorphic as an A-module to the homogeneous part of degree n of the symmetric A-algebra on $L^A\Omega^1_A$:

$$P_nA \cong S^A_n L^A_+ \Omega^1_A \quad (n \geqslant 0). \tag{15}$$

However, the proof of this assertion in [2] has a gap, as the isomorphism given there is not well-defined. Indeed, the map in question sends the element

$$P_nA \ni b\bigl\{a_0, \bigl\{a_1, \ldots, \{a_{n-1}, a_n\}\ldots\bigr\}\bigr\} \quad (a_i \in A)$$

to the element

$$b\bigl[da_0, \bigl[da_1, \ldots [da_{n-1}, da_n]\ldots\bigr]\bigr] \in S^A_n L^A_+ \Omega^1_A.$$

However a calculation shows that this rule maps

$$0 = \bigl\{a_1, a_3\{a_2, a_4\}\bigr\} + \bigl\{a_1, a_2\{a_3, a_4\}\bigr\} - \bigl\{a_1, \{a_2a_3, a_4\}\bigr\}$$

to the element

$$[da_1, da_3][da_2, da_4] + [da_1, da_2][da_3, da_4],$$

which is nonzero in general. I do not know whether the isomorphism (15) still holds for every smooth algebra A. It certainly holds for symmetric algebras, as is immediate from (12). We show in Theorem 1.4 below that it also holds for local smooth algebras. For a weaker version of (15) which holds globally, see Theorem 1.6. The following lemma is well-known.

Lemma 1.3. *Let X be a set, $V = \bigoplus_{x\in X} kx$ the free vectorspace on X. Then the set*

$$Y := X \cup \bigcup_{n=1}^{\infty} \bigl\{ \bigl[x_1, \bigl[x_2, \bigl[\dots, [x_n, x_{n+1}]\dots\bigr]\bigr]\bigr]\colon x_i \in X \bigr\}$$

generates LV as a vectorspace. In particular, there is a basis Z of LV such that $X \subset Z \subset Y$.

Proof. Straightforward induction. □

Theorem 1.4. *Let A be a local smooth algebra. Then A satisfies* (15).

Proof. Let $x_1, \dots, x_n \in A$ be a regular system of parameters and $V = \bigoplus_{i=1}^{n} k\,dx_i \subset \Omega^1_A$. We have

$$S^A L^A_+ \Omega^1_A = A \otimes SL_+ V.$$

Put $L = LV$; then

$$\Omega^1_A = A \otimes V, \qquad \Omega^1_{SL_+} = SL_+ \otimes L_+, \qquad \Omega^1_{A\otimes SL_+} = A \otimes SL_+ \otimes L. \tag{16}$$

Consider the permutation isomorphism

$$\tau : \Omega^1_A \otimes SL_+ = A \otimes V \otimes SL_+ \cong A \otimes SL_+ \otimes V.$$

Under the identifications (16) the de Rham derivation $d_{A\otimes SL_+}$ is identified with

$$D := \tau \circ (d_A \otimes \mathrm{id}_{SL_+}) + \mathrm{id}_A \otimes d_{SL_+}.$$

Put

$$\phi = \mathrm{id}_{A\otimes SL_+} \otimes [,] : A \otimes SL_+ \otimes \Lambda^2 L \to A \otimes SL_+.$$

One checks that

$$\{p, q\} := \phi(Dp \wedge Dq)$$

is a Poisson bracket. Note that for $1 \leqslant i_1, \dots, i_{r+1} \leqslant n$, we have

$$\bigl\{x_{i_1}, \bigl\{x_{i_2}, \dots, \{x_{i_r}, x_{i_{r+1}}\}\dots\bigr\}\bigr\} = \bigl[dx_{i_1}, \bigl[dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}]\dots\bigr]\bigr].$$

It suffices to show that the Poisson algebra $(A \otimes SL_+, \{,\})$ together with the inclusion $A = A \otimes k = A \otimes S_0 L_+ \subset A \otimes SL_+$ has the universal property of PA. Let P be a Poisson algebra and $f : A \to P$ a homomorphism of commutative algebras. Write $p_i = f(x_i)$. By Lemma 1.3, we may extend $B_0 = \{dx_1, \dots, dx_n\}$ to a homogeneous basis B of L such that every element of $B' := B \backslash B_0$ be of the form $[dx_{i_1}, [dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}]] \dots]$ $(r \geqslant 1)$. View P as an A-module via f and consider the A-module homomorphism $\theta : L_+^A \Omega_A^1 = A \otimes L_+ \to P$ defined on elements of B' by

$$\theta\big[dx_{i_1}, \big[dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}] \dots\big]\big] = \big\{p_{i_1}, \big\{p_{i_2}, \dots, \{p_{i_r}, p_{i_{r+1}}\} \dots\big\}\big\}. \tag{17}$$

Note that, as defined,

$$\theta[l_1, l_2] = \{\theta l_1, \theta l_2\} \quad (l_1, l_2 \in B'). \tag{18}$$

Indeed by Lemma 1.3, the two sides of this identity are defined by the same linear combination of the elements of B' and of their images. The prescription (17) together with the prescription that θ extend f, determine a unique map $\theta : A \otimes SL_+ \to P$ which satisfies (18) for $l_1, l_2 \in B'' := \{x_1, \dots, x_n\} \cup B'$. It follows that θ is a Poisson homomorphism; uniqueness is clear. $\square$

1.5. A weaker version of property (15)

Let V be a vectorspace. Combining the two gradings (7), (8) we obtain a bigrading

$$SLV = \bigoplus_{p,q \geqslant 0} S_q^p LV, \tag{19}$$

where

$$S_q^p LV := S^p LV \cap S_q LV.$$

Now let A be a commutative algebra, and consider the projection

$$\pi : SLA \twoheadrightarrow PA. \tag{20}$$

The map π is homogeneous for the $\|_*$-degree but not for the $\|^*$-degree. However the ideal

$$\mathcal{H}^n := \pi\left(\bigoplus_{p \geqslant n} S^p LA\right) \quad (n \geqslant 0)$$

is homogeneous with respect to $\|_*$ and therefore the graded ring $GPA = G_{\mathcal{H}} PA$ is actually bigraded

$$GPA = \bigoplus_{p,q \geqslant 0} G_q^p PA.$$

On the other hand, (19) carries over to the free Lie algebra of any A-bimodule. In particular, $S^A L^A \Omega_A^1$ is a bigraded algebra.

Theorem 1.6. *Let A be a commutative algebra. Then for the bigraded structures defined in* 1.5*, there is a natural surjective homomorphism of bigraded algebras*

$$\phi : S^A L^A \Omega_A^1 \twoheadrightarrow GPA.$$

If furthermore A is smooth, then ϕ is an isomorphism.

Proof. The map (20) is the quotient by $\langle\!\langle IA \rangle\!\rangle$, whence it factors through a map

$$\overline{\pi} : A \otimes SL_+A = \frac{SLA}{\langle IA \rangle} \twoheadrightarrow PA.$$

In particular, $\overline{\pi}$ induces a homogeneous, surjective homomorphism of graded A-modules

$$p : A \otimes L_+A \xrightarrow{\overline{\pi}} \mathcal{H}^1 \twoheadrightarrow G_*^1 PA.$$

Write $\rho : A \to SA$ for the canonical inclusion. The ideal IA is generated by the elements

$$u(a,b) := \rho(ab) - \rho a \rho b \quad (a, b \in A).$$

Let $h_i(a_1, \dots, a_n; b, c)$ be the homogeneous part of $\|^*$-degree one of the element $g_i(\rho a_1, \dots, \rho a_n; u(b,c))$ of (10). By (11), the elements $h_i(a_1, \dots, a_n; b, c)$, $1 \leqslant i \leqslant n$, $a_i, b, c \in A$ generate $\ker p$ as an A-module. A calculation shows that

$$\begin{aligned}
&h_i(a_1, \dots, a_n; b, c) \\
&\quad = 1 \otimes \bigl\{\rho a_1, \dots, \bigl\{\rho a_i, \bigl\{\rho(bc), \bigl\{\rho a_{i+1}, \dots, \{\rho a_{n-1}, \rho a_n\} \dots\bigr\}\bigr\}\bigr\} \dots\bigr\} \\
&\qquad - \rho b \otimes \bigl\{\rho a_1, \dots, \bigl\{\rho a_i, \bigl\{\rho c, \bigl\{\rho a_{i+1}, \dots, \{\rho a_{n-1}, \rho a_n\} \dots\bigr\}\bigr\}\bigr\} \dots\bigr\} \\
&\qquad - \rho c \otimes \bigl\{\rho a_1, \dots, \bigl\{\rho a_i, \bigl\{\rho b, \bigl\{\rho a_{i+1}, \dots, \{\rho a_{n-1}, \rho a_n\} \dots\bigr\}\bigr\}\bigr\} \dots\bigr\}.
\end{aligned}$$

It follows from this that

$$M := \ker\bigl(A \otimes A \twoheadrightarrow \Omega_A^1\bigr) \oplus \ker p$$

is the Lie ideal generated by $\ker(A \otimes A \twoheadrightarrow \Omega_A^1)$ in the A-Lie algebra $A \otimes LA$. Hence $(A \otimes LA)/M = L^A \Omega_A^1$, and

$$L_+^A \Omega_A^1 \cong G^1 PA. \tag{21}$$

The map ϕ of the theorem is that induced by (21); it is surjective because $G^1 PA$ generates GPA as an A-algebra. Assume now that A is smooth; we must prove that ϕ is injective. This is a local question, so we may further assume that A is local. Let $x_1, \dots, x_n$ be a regular system of parameters. By the proof of Theorem 1.4 there is an isomorphism $\psi : PA \xrightarrow{\sim} S^A L_+^A \Omega_A^1$ such that

$$\psi\bigl\{x_{i_1}, \bigl\{x_{i_2}, \dots, \{x_{i_r}, x_{i_{r+1}}\} \dots\bigr\}\bigr\} = \bigl[dx_{i_1}, \bigl[dx_{i_2}, \dots, [dx_{i_r}, dx_{i_{r+1}}] \dots\bigr]\bigr]. \tag{22}$$

Furthermore, the induced map $G\psi : GPA \to S^A L_+^A \Omega_A^1$ still verifies (22). Thus $G\psi \circ \phi$ is the identity map, because it is so on generators. In particular, ϕ is injective. □

2. Local quantization of the Poisson envelope

2.0. PBW quantization

Let $\mathfrak{g}$ be a Lie algebra, $S\mathfrak{g}$ and $U\mathfrak{g}$ the symmetric and universal enveloping algebras, and consider the symmetrization map

$$e : S\mathfrak{g} \to U\mathfrak{g}, \qquad e(g_1 \dots g_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} g_{\sigma(1)} \dots g_{\sigma(n)}. \tag{23}$$

By the Poincaré–Birkhoff–Witt theorem, the associative product

$$B : S\mathfrak{g} \otimes S\mathfrak{g} \to S\mathfrak{g}, \qquad B(x \otimes y) = e^{-1}(exey)$$

decomposes as a sum

$$B = \sum_{p=0}^{\infty} B_p \quad \text{where } B_p\big(S^n\mathfrak{g}\big) \subset S^{n-p}\mathfrak{g}\ (n, p \geqslant 0). \tag{24}$$

We have $B_0(x \otimes y) = xy$, $B_1(x \otimes y) = \frac{1}{2}\{x, y\}$. Explicit formulas for all the B_p are given in [1]. It also proved in loc. cit. that for each $p \geqslant 0$, B_p is a differential operator of order $\leqslant p$. We call the map

$$B(\hbar) := \sum_{n \geqslant 0} B_n \hbar^n : S\mathfrak{g} \otimes S\mathfrak{g}[\![\hbar]\!] \to S\mathfrak{g}[\![\hbar]\!]$$

the PBW *quantization*. The next lemma establishes the properties of the product B with respect to the commutator filtration (1) in the case when $\mathfrak{g}$ is free.

Lemma 2.1. *Let V be a vectorspace, LV the free Lie algebra, TV the tensor algebra, and e as in* (23). *Then*

(1) $e(\mathrm{Poiss}_{\geqslant n} V) = F_n TV$.
(2) *The operator B_p* : Poiss $V \otimes$ Poiss $V \to$ Poiss V *of* (24) *is homogeneous of degree $+p$ for the $\|_*$-degree* (7).

Proof. One checks that if $\mathfrak{g}$ is any Lie algebra and $U\mathfrak{g}$ its enveloping algebra, then for $F_0\mathfrak{g} = \mathfrak{g}$, $F_d\mathfrak{g} = [\mathfrak{g}, F_{d-1}\mathfrak{g}]$ $(d \geqslant 1)$ we have

$$F_n U\mathfrak{g} = \sum_r \sum_{d_1 + \dots + d_r \geqslant n} F_{d_1}\mathfrak{g} \cdot \dots \cdot F_{d_r}\mathfrak{g}.$$

For $\mathfrak{g} = LV$, we obtain

$$F_nTV = \sum_r \sum_{d_1+\ldots+d_r \geqslant n} L_{d_1}V \cdot \cdots \cdot L_{d_r}V. \tag{25}$$

From (25) and (9) it is clear that $e(\mathrm{Poiss}_{\geqslant n} V) \subset F_nTV$. We must show that $F_nTV \subset e(\mathrm{Poiss}_{\geqslant n} V)$. Consider the following subspace of F^nTV:

$$\mathcal{A}_{p,n} := \sum_{r \leqslant p} \sum_{d_1+\cdots+d_r \geqslant n} L_{d_1}V \cdot \cdots \cdot L_{d_r}V.$$

Clearly $F_nTV = \bigcup_{p \geqslant 1} \mathcal{A}_{p,n}$. An inductive argument similar to that of the usual proof of the surjectivity of e shows that for $p \geqslant 1$, $\mathcal{A}_{p,n} \subset e(\mathrm{Poiss}_{\geqslant n} V)$. This proves assertion (1). Assertion (2) follows by counting degrees in the formula for B_p given in [1, 1.1]. $\square$

Corollary 2.2 (Compare [2, 3.4.7]). *The natural map*

$$\mathrm{Poiss}\, V = PSV \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} F_nTV/F_{n+1}TV$$

is an isomorphism.

2.3. PBW quantization of PA for A local and smooth

Let $(A, \mathcal{M})$ be a smooth local commutative algebra and $X = \{x_1, \ldots, x_n\} \subset \mathcal{M}$ a regular system of parameters. Set $V = k^n$. We are going to combine Theorem 1.4 and the PBW quantization of Poiss V to obtain an associative product

$$B^X(\hbar) = \sum_{p=0}^{\infty} B_p^X \hbar^p : PA \otimes PA[\![\hbar]\!] \to PA[\![\hbar]\!].$$

Because the map $B_p : \mathrm{Poiss}\, V \otimes \mathrm{Poiss}\, V \to \mathrm{Poiss}\, V$ is a differential operator, it is continuous with respect to the topology of any ideal $I \subset \mathrm{Poiss}\, V$. Applying this for $I = V \cdot \mathrm{Poiss}\, V$ and completing we obtain the horizontal solid arrow in the following commutative diagram:

$$\begin{tikzcd} PA \otimes PA \arrow[r, dotted, "B_p^X"] \arrow[d, "\iota_X \widehat{\otimes} \iota_X"'] & PA \arrow[d, "\iota_X"] \\ k[\![t]\!] \widehat{\otimes} SL_+V \widehat{\otimes} k[\![t]\!] \widehat{\otimes} SL_+V \arrow[r, "\hat{B}_p"'] & k[\![t]\!] \widehat{\otimes} SL_+V. \end{tikzcd} \tag{26}$$

Here $\widehat{\otimes}$ is the completed tensor product and $k[\![t]\!]$ is shorthand for $k[\![t_1,\dots,t_n]\!]$. The map ι_X is the composite

$$\iota_X : PA \xrightarrow[\sim]{\alpha_X} A\otimes SL_+V \hookrightarrow \hat{A}\,\widehat{\otimes}\,SL_+V \xrightarrow[\sim]{j_X\otimes 1} k[\![t]\!]\,\widehat{\otimes}\,SL_+V,$$

where α_X is the isomorphism of 1.4, $\hookrightarrow$ is the passage to completion and $j_X : \hat{A}\cong k[\![t]\!]$ is the isomorphism determined by $x_i\mapsto t_i$ $(i=1,\dots,n)$. Because $A\hookrightarrow\hat{A}$ is injective, so are both vertical maps in (26). The map B_p^X is defined by the following lemma.

Lemma 2.4. *The map $\widehat{B}_p$ of* (26) *sends the image of $\iota_X\,\widehat{\otimes}\,\iota_X$ to the image of ι_X.*

Proof. Let $Z=X\cup\{y_1,y_2,\dots\}$ be a basis as that of Lemma 1.3. Every monomial on the elements of Z can be written as $x^{\alpha'}y^{\alpha''}=\prod_{i=1}^n x_i^{\alpha'(i)}\prod_{j=1}^\infty y_j^{\alpha''(j)}$ for some multi-indices $\alpha':\{1,\dots,n\}\to\mathbb{Z}_{\geqslant 0}$ and $\alpha'':\mathbb{Z}_{\geqslant 1}\to\mathbb{Z}_{\geqslant 0}$ with $\alpha''(n)=0$ for $n\gg 0$. Let $\frac{\partial^{|\alpha|}}{\partial x^{\alpha'}\partial y^{\alpha''}}$ be the higher derivation with symbol $x^{\alpha'}y^{\alpha''}$. Because $B_p : SLV\otimes SLV\to SLV$ is a bidifferential operator of order $\leqslant p$, it can be written as an SLV-linear combination of cup products of higher derivations with respect to the basis Z

$$B_p=\sum_{|\alpha|,|\beta|\leqslant p} c_{\alpha,\beta}\frac{\partial^{|\alpha|}}{\partial x^{\alpha'}\partial y^{\alpha''}}\cup\frac{\partial^{|\beta|}}{\partial x^{\beta'}\partial y^{\beta''}}.$$

Since each of the higher derivations above maps $A\subset k[\![t]\!]$ to itself, so does $\widehat{B}_p$. □

Theorem 2.5. *Let $(A,\mathcal{M})$ be a smooth local algebra, $X\subset\mathcal{M}$ a regular system of parameters, $p\geqslant 0$, B_p^X as in* (26) *and $\widehat{PA}=\varprojlim_d PA/P_{>d}A=\prod_{d=0}^\infty P_dA$. Then*

(1) *$B^X(\hbar)=\sum_{p=0}^\infty B_p^X\hbar^p : PA\otimes PA[\![\hbar]\!]\to PA[\![\hbar]\!]$ is associative.*
(2) *B_p^X is a differential operator of order $\leqslant p$.*
(3) *$B_p^X(P_nA\otimes P_mA)\subset P_{n+m+p}A$.*
(4) *The map $B^X(\hbar)$ induces a continuous associative product*

$$B^X(1):=\sum_{p=0}^\infty B_p^X : \widehat{PA}\,\widehat{\otimes}\,\widehat{PA}\to\widehat{PA}.$$

(5) *For the associative algebra $Q_X=(\widehat{PA},B^X(1))$, we have*

$$F_nQ_X=\prod_{d\geqslant n}P_dA.$$

(6) *There is a commutative diagram of monomorphisms*

$$\begin{tikzcd} k\{t_1,\dots,t_n\} \arrow[r] \arrow[dr, "\mathrm{inc}"'] & Q_X \arrow[d] \\ & k\{\{t_1,\dots,t_n\}\}, \end{tikzcd}$$

where inc *is the canonical inclusion of the noncommutative polynomials into the noncommutative power series.*

Proof. Assertions (1)–(3) are immediate from the analogous properties of the PBW quantization; (4) follows from (3), and (5) from Lemma 2.1. It is clear from the definition of Q_X that there is a diagram as that in (5) but with $\widehat{SLV} := (\prod_{n\geqslant 0} k[\![t]\!] \mathbin{\widehat{\otimes}} SL_+V, \widehat{B})$ substituted for $k\{\{t\}\}$. Note that, for $I = \sum t_i \cdot k[t_1,\dots,t_n]$, $\widehat{SLV}$ is the completion of SLV with respect to the filtration $\{I\cdot SLV + S_{\geqslant n}LV\colon n \geqslant 0\}$, and that $J := e^{-1}(\langle t_1,\dots,t_n\rangle) = I \oplus S_+LV$. By Lemmas 2.1 and 2.6, $\widehat{SLV} \cong \varprojlim_n TV/e(J)^n = k\{\{t_1,\dots,t_n\}\}$. $\square$

Lemma 2.6. *Let $G = \bigoplus_{n=0}^{\infty} G_n$ be a graded commutative algebra. Assume G is additionally equipped with an associative—but not necessarily commutative—product*

$$\Phi = \sum_{p=0}^{\infty} \Phi_p : G\otimes G \to G$$

such that Φ_0 is the original commutative product, and that for each $p \geqslant 1$, Φ_p is a bidifferential operator. Let $I \subset G_0$ be an ideal for Φ_0, and $J \subset G$ the Φ-ideal it generates. Then the linear topologies induced on the underlying vectorspace of G by the filtrations $\{J^n + G_{\geqslant n}\colon n \geqslant 0\}$ and $\{I^nG + G_{\geqslant n}\colon n \geqslant 0\}$ coincide.

Proof. It suffices to prove that for each $d \geqslant 0$ the filtrations $\{J^n + G_{\geqslant d+1}/G_{\geqslant d+1}\colon n \geqslant 0\}$ and $\{I^nG + G_{\geqslant d+1}/G_{\geqslant d+1}\colon n \geqslant 0\}$ of $G/G_{\geqslant d+1}$ are equivalent. Thus we may assume that $G_m = 0$ for $m \geqslant d+1$. Hence Φ is a bidifferential operator. Let α be the order of Φ. We write $x \star y := \Phi(x\otimes y)$, and if $X \subset G$ is any subspace, we put $X^{\star n}$ for the subspace generated by all products $x_1 \star\cdots\star x_n$ with $x_i \in X$. Let $i \in I$, $p \geqslant 1$, $n, r \geqslant 0$, $F_i(x) := i \star x$. Because F_i is a differential operator of order $\leqslant \alpha$,

$$F_i\big(I^{p\alpha+r}G_n\big) \subset I^{p\alpha+r+1}G_n + I^pG_{\geqslant n+1}. \tag{27}$$

Using (27), one checks by induction that for $(c_r\dots c_0) := \sum_{i=0}^r c_i\alpha^i$,

$$I^{\star(c_r\dots c_0)} \subset M_{(c_r\dots c_0)} := \sum_{j=0}^{r} I^{(c_r\dots c_j)}G_{\geqslant j} + G_{\geqslant r+1}. \tag{28}$$

We remark that $M_{(c_r \dots c_0)}$ is an ideal for both $\star$ and the original product. Hence for $N \geqslant (c_r \dots c_0) + d$ and $r \geqslant d$,

$$J^{\star N} \subset (I \oplus G_{\geqslant 1})^{\star N} \subset \langle I^{\star(c_r \dots c_0)} \rangle_\star \subset M_{(c_r \dots c_0)} \subset I^{(c_r \dots c_d)} G,$$

where the subindex $_\star$ denotes two sided ideal generated by the product $\star$. Now using (28), and noting that $I^{\star n} \star G_d = I^n G_d$ and that in general the projection $G \twoheadrightarrow G_j$ maps $I^{\star n} \star G_j$ surjectively onto $I^n G_j$, one checks that, for $r \geqslant d$,

$$M_{(c_r \dots c_0)} = \sum_{j=0}^{d} I^{\star(c_r \dots c_j)} \star G_{d-j}.$$

It follows that

$$J^{\star n} \supset M_{n\alpha^d} \supset I^{n\alpha^d}. \qquad \square$$

3. Smooth nilcommutative and nil-Poisson algebras

3.0. Nil-Poisson algebras

Let P be a Poisson algebra. Put $F_0 P = P$ and inductively

$$F_{n+1} P := \sum_{i=1}^{n} F_i P F_{n+1-i} P + \sum_{i=0}^{n} \langle \{F_i P, F_{n-i} P\} \rangle \tag{29}$$

for $n \geqslant 0$. This is the Poisson analogue of the commutator filtration (1). For example, if A is any commutative algebra, then

$$F_r P A = P_{\geqslant r} A = \bigoplus_{n=r}^{\infty} P_n A \quad (r \geqslant 0).$$

The analogue of a nilcommutative algebra of order $\leqslant d$ is called a *nil-Poisson* algebra of order $\leqslant d$. The category of nil-Poisson algebras of order $\leqslant d$ is NP_d. We put $NP = \bigcup_{d \geqslant 0} NP_d$. Formal d-smoothness and d-smoothness for objects of NP_d are the obvious analogues of the same properties for objects of NC_d as defined in Section 1. If A is (formally) smooth in the commutative sense, then $PA/P_{>d}A$ is (formally) d-smooth. It turns out that every (formally) d-smooth Poisson algebra is of this form; see Proposition 3.3. The following lemma is the analogue of [2, 1.2.7] for Poisson algebras.

Lemma 3.1. *Let $P \in NP_d$ and $f : P \to P$ a Poisson endomorphism. Assume the induced map $P/F_1 P \to P/F_1 P$ is the identity. Then the restriction of f to $F_d P$ is the identity also.*

Proof. Consider the map $D : P \to P$, $Dp := fp - p$. We have

$$D\{p,q\} = \{Dp,q\} + \{p,Dq\} + \{Dp,Dq\},$$
$$D(pq) = pDq + qDp + DpDq. \tag{30}$$

By hypothesis, $DP \subset F_1P$; it follows from this, using (30), (29) and induction, that for $n \geqslant 0$, $D(F_nP) \subset F_{n+1}P$. In particular, $D(F_dP) = 0$. □

Remark 3.2. The proof of the lemma above still applies if one substitutes NC_d for NP_d and "algebra endomorphism" for "Poisson endomorphism." This gives an alternate proof of [2, 1.2.7].

Proposition 3.3. *Let $P \in NP_d$, $A = P/F_1P$. Then the following conditions are equivalent*:

(i) *P is d-formally smooth.*
(ii) *A is* 0*-formally smooth and $PA/P_{>d}A \cong P$.*

The same holds if we replace "formally smooth" by "smooth" in both (i) *and* (ii).

Proof. That (ii) ⇒ (i) is clear, as is that (i) implies A is formally smooth. Use the formal smoothness of A to obtain a section $s : A \to P \in (\!(\mathrm{Comm})\!)$ of the projection $P \twoheadrightarrow A$, and then the universal property of PA to lift s to a map of extensions $\alpha : PA/P_{>d}A \to P$. To prove that if (i) holds then $\alpha : PA/P_{>d}A \to P$ is an isomorphism, note that, by the hypothesis on P, there is a map $\beta : P \to PA/P_{>d}A$ which descends to the identity of A. One is thus reduced to showing that $\alpha\beta$ and $\beta\alpha$ are isomorphisms. This follows from Lemma 3.1. □

Part (1) ⇒ (2) of the following theorem is due to Kapranov [2, 4.2.1]; we give a new proof.

Theorem 3.4. *Let $R \in NC_d$, $G = \bigoplus_{n=0}^{d} G_n$ the associated graded Poisson algebra, $A = G_0$, $\pi : R \to A$ the projection. The following conditions are equivalent*:

(1) *R is d-smooth.*
(2) *A is* 0*-smooth and the canonical map $PA/P_{>d}A \to G$ is an isomorphism.*
(3) *For every maximal ideal $\mathcal{M} \subset A$, there is a regular system of parameters $X \subset \mathcal{M} \cdot A_{\mathcal{M}}$ such that for Q_X as in Theorem* 2.5*, the Øre localization of R at $\pi^{-1}(\mathcal{M})$ is isomorphic to $Q_X/F_{>d}Q_X$.*

Proof. Assume (1) holds. Then clearly A is 0-smooth. Let $\mathcal{M} \subset A$ be a maximal ideal, $\mathbf{M} = \pi^{-1}(\mathcal{M})$, $X \subset \mathcal{M} \cdot A_{\mathcal{M}}$ a regular system of parameters and $Q = Q_X/F_{>d}Q_X$. Because the Øre localization $R_{\mathbf{M}}$ is d-smooth, the identity of $A_{\mathcal{M}}$ can be lifted to a map $f : R_{\mathbf{M}} \to Q$. By Lemma 3.1, the map induced by f at the graded level is an isomorphism, whence f is an isomorphism. We have just proved that (1) ⇒ (3). It is clear that (3) ⇒ (2).

We prove next that (2) $\Rightarrow$ (1), by induction on $d \geqslant 0$. The case $d = 0$ is tautological. Assume $d \geqslant 1$ and that the theorem is true for $d - 1$. Let $R \in NC_d$ satisfy the hypothesis of the theorem. Then by inductive assumption $R_{d-1} = R/F_d R$ is $(d-1)$-smooth. Let $\pi : U \twoheadrightarrow R_{d-1}$ be the universal central extension as defined in [2, 1.3.6]. By [2, 1.6.2], U is d-smooth. One checks that $\ker \pi = F_d U$. By [2, 1.3.8], there is a map $\alpha : U \to R$ which induces the identity of R_{d-1}. Consider the Poisson homomorphism β induced by α at the associated graded level. Because (1) $\Rightarrow$ (2), β is an endomorphism of $PA/P_{>d}A$. By virtue of Lemma 3.1, because β induces the identity modulo $P_d A$ and is homogeneous, it has to be the identity. Thus α is an isomorphism. $\square$

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